

ram p. kanwal

LINEAR INTEGRAL EQUATIONS

**theory
and technique**

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Theory and Technique

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Linear Integral Equations

Theory and Technique

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Pennsylvania State University

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ACADEMIC PRESS

1971

New York and London

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ACADEMIC PRESS, INC.

111 Fifth Avenue, New York, New York 10003

United Kingdom Edition published by
ACADEMIC PRESS, INC. (LONDON) LTD.
Berkeley Square House, London W1X 6BA

LIBRARY OF CONGRESS CATALOG CARD NUMBER: 77-156268

AMS (MOS) 1970 Subject Classification: 45A05, 45B05, 45C05,
45D05, 45E05, 45E10, 45E99, 45F05, 35C15, 44A25, 44A35, 47G05

PRINTED IN THE UNITED STATES OF AMERICA

IN LOVING MEMORY OF
MY GRANDMOTHER

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PREFACE

Many physical problems which are usually solved by differential equation methods can be solved more effectively by integral equation methods. Indeed, the latter have been appearing in current literature with increasing frequency and have provided solutions to problems heretofore not solvable by standard methods of differential equations. Such problems abound in many applied fields, and the types of solutions explored here will be useful particularly in applied mathematics, theoretical mechanics, and mathematical physics.

Each section of the book contains a selection of examples based on the technique of that section, thus making it possible to use the book as the text for a beginning graduate course. The latter part of the book will be equally useful to research workers who encounter boundary value problems. The level of mathematical knowledge required of the reader is no more than that taught in undergraduate applied mathematics courses. Although no attempt has been made to attain a high level of mathematical rigor, the regularity conditions for every theorem have been stated precisely. To keep the book to a manageable size, a few long proofs have been omitted. They are mostly those proofs which do not appear to be essential in the study of the subject for purposes of applications.

We have omitted topics such as Wiener-Hopf technique and dual integral equations because most of the problems which can be solved by these methods can be solved more easily by the integral equation methods presented in Chapters 10 and 11. Furthermore, we have concentrated mainly on three-dimensional problems. Once the methods have been grasped, the student can readily solve the corresponding plane problems. Besides, the powerful tools of complex variables are available for the latter problems.

The book has developed from courses of lectures on the subject given by the author over a period of years at Pennsylvania State University.

Since it is intended mainly as a textbook we have omitted references to the current research literature. The bibliography contains references to books which the author has consulted and to which the reader is referred occasionally in the text. Chapter 10 is based on author's joint article with Dr. D. L. Jain (*SIAM Journal of Applied Mathematics*, **20**, 1971) while Chapter 11 is based on an article by the author (*Journal of Mathematics and Mechanics*, **19**, 1970, 625–656). The permission of the publishers of these two journals is thankfully acknowledged. These articles contain references to the current research literature pertaining to these chapters. The most important of these references are the research works of Professor W. E. Williams and Professor B. Noble.

Finally, I would like to express my gratitude to my students Mr. P. Gressis, Mr. A. S. Ibrahim, Dr. D. L. Jain, and Mr. B. K. Sachdeva for their assistance in the preparation of the textual material, to Mrs. Suzie Mostoller for her pertinent typing of the manuscript, and to the staff of Academic Press for their helpful cooperation.

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1.1. DEFINITION

An integral equation is an equation in which an unknown function appears under one or more integral signs. Naturally, in such an equation there can occur other terms as well. For example, for $a \leq s \leq b$, $a \leq t \leq b$, the equations

$$f(s) = \int_a^b K(s, t) g(t) dt, \quad (1)$$

$$g(s) = f(s) + \int_a^b K(s, t) g(t) dt, \quad (2)$$

$$g(s) = \int_a^b K(s, t) [g(t)]^2 dt, \quad (3)$$

where the function $g(s)$ is the unknown function while all the other functions are known, are integral equations. These functions may be complex-valued functions of the real variables s and t .

Integral equations occur naturally in many fields of mechanics and mathematical physics. They also arise as representation formulas for the solutions of differential equations. Indeed, a differential equation can be